Majorisations for the eigenvectors of graph-adjacency matrices

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We develop majorisation results that characterise changes in eigenvector components of a graph’s adjacency matrix when its topology is changed. Specifically, for general (weighted, directed) graphs, we characterise changes in dominant eigenvector components for single- and multi-row incrementations. We also show that topology changes can be tailored to set ratios between the components of the dominant eigenvector. For more limited graph classes (specifically, undirected, and reversibly-structured ones), majorisations for components of the subdominant and other eigenvectors upon graph modifications are also obtained.

Keywords: algebraic graph theory; network design; dynamical networks; adjacency matrix; eigenvectors; majorisation

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1. Introduction

During the last 20 years or so, new research at the interface of dynamical systems, graph theory, and linear algebra has provided significant understanding of the relationship between a network’s topology and its dynamics. However, there is an increasing need to not only characterise but also design the dynamics of complex networks, in several domains. For instance, in epidemic control, management actions that change a network topology of disease spread (e.g. targeted restriction on traveling, or quarantine) may need to be designed, to eliminate the spread (Preciado, Zargham, Enyioha, Jadbabaie, & Pappas, 2013; Wan, Roy, & Saberi, 2008) with limited resources. Similarly, in sensor networking applications, tools for designing the sensor-to-sensor communication topology are needed to shorten the duration for the sensors to complete algorithmic tasks (e.g. reach a consensus) and therefore to reduce energy consumption (e.g. Wan, Namuduri, Akula, & Varanasi, 2012; Wan, Roy, & Saberi, 2008; Xiao & Boyd, 2004). While there are some nascent efforts on designing a network’s dynamics, much remains to be done – especially in the realistic circumstance that a network’s topology can only be partially designed or modified. While the details of network modification/design are application-specific, it is generally true that partial modifications of a network’s structure affect complex changes in its dynamics that are not well understood and require new methods for characterisation.

One promising route for understanding the impact of network modifications on dynamics is to identify how the topology changes impact the spectra (eigenvalues and eigenvectors) of certain matrices defined from the network’s topology (graph), such as the Laplacian or adjacency matrices: these matrices often define core network dynamics, and hence characterising their spectra gives an insight into the dynamics. Spectral properties of the adjacency and Laplacian matrices have been characterised in both the Algebraic Graph Theory and the non-negative matrices’ literature (e.g. Berman, Neumann, & Stern, 1989; Berman & Plemmons, 1979; Chung, 1997); a bulk of this literature is focused on characterising eigenvalues rather than eigenvectors of the graph matrices, but some graph-based characterisations of eigenvectors are also available (Fiedler, 1975; Stewart & Sun, 1990). While many spectral characterisations have been developed, only a few results specify the impact of topology changes on adjacency/Laplacian eigenvalues, and eigenvector components, especially for asymmetrical (directed) graphs. Characterisation of such impacts forms a crucial step in the study of networked dynamical systems.

In this paper, we investigate the impact of topology changes on the eigenvectors of the adjacency matrix of a network’s graph. Our study is aligned with Merris (1998), in which eigenvectors of Laplacian matrices of unweighted and undirected graphs are studied. In particular, the article Merris (1998) determined the effect of adding, deleting, or contracting edges in the graph on the subdominant eigenvector of the graph’s Laplacian. The results we develop here complement those presented in Merris (1998). However, we consider weighted undirected as well as weighted directed
graphs, and consider different classes of topology modifications as compared to Merris (1998). Specifically, our focus here is on developing majorisations (comparisons) for the eigenvector components of the adjacency matrices of both directed and undirected graphs, upon structured modification of edge weights in the graph. We consider several types of topology changes and graph topologies classes, and develop majorisations for extremal and/or sub-dominant eigenvalues and associated eigenvectors in these cases.

The eigenvector majorisation results that we develop rely critically on (1) classical results on non-negative matrices’ eigenvalues (Berman & Plemmons, 1979), and (2) matrix perturbation concepts. From one viewpoint, our result extend existing eigenvalue majorisations for non-negative matrices (Berman & Plemmons, 1979), toward comparative characterisations of eigenvectors (see also Roy, Saberi, & Wan, 2008). From another viewpoint, our results can be viewed as providing tighter bounds on eigenvector components than would be obtained from perturbation arguments (see Stewart & Sun, 1990), for the special class of non-negative matrices.

While very few studies explicitly consider eigenvector-component majorisation, we have recently become aware of two characterisations of non-negative matrices (Sahi, 1993 and Sahi, 2010), that are closely aligned with our work. In Sahi (2010), the author developed majorisations for entries of the resolvent of a non-negative matrix, when a given entry of the matrix is arbitrarily reduced. In Sahi (1993), a special type of convexity for the dominant eigenvector components of substochastic matrices is presented. The results we develop in this paper can be transformed into results in Sahi (2010) and vice versa in some special cases, although in general the results are not equivalent. Nevertheless, our results are in the same vein as the ones presented in Sahi (2010) and Sahi (1993).

In brief, the contributions made in this article relative to the literature are the following:

- For two structured graph classes (weighted undirected and reversible graphs), we show how ratios between eigenvector components of graph-adjacency matrices behave under special structured changes to the graph topologies. These results are similar in spirit to Merris (1998), but encompass broader graph classes and different structural perturbations.
- For generic weighted directed graphs, we show how ratios between the components of dominant eigenvectors of graph-adjacency matrices behave under arbitrary changes to graph topologies. Several different classes of graph modifications are considered, including simultaneous modifications of multiple edges.
- Motivated by design needs, we show that topology changes can be constructed that achieve a specific ratio requirements for dominant eigenvector components of graph-adjacency matrices for generic graph classes.
- Via a conceptual example on shaping spread processes, we illustrate how the obtained majorisations can inform dynamical-network analysis and design.

The remainder of the article is organised as follows. In Section 2, we motivate and formulate the study of eigenvector properties for adjacency matrices of graphs of three types: symmetric (or undirected), reversible (in the sense of Markov-chain reversibility), and arbitrary undirected graphs. We also define graph modifications of interest for these graph classes. In Section 3, eigenvector majorisation results are provided for the case of symmetric weight increments in an undirected graph, wherein edge weight between a pair of vertices is incremented. In Section 4, we consider balanced weight increments in a reversible graph, wherein weights of directed edges between a pair of distinct vertices are increased in a specific way. In Section 5, we consider a more general case where edge weights in an arbitrary directed graph are increased in complex ways, but focus on majorisations for the dominant eigenvector only. A conceptual example illustrating use of the majorisations is also included.

## 2. Problem formulation

We consider a weighted and directed graph with the $n$ vertices, $V_1, V_2, \ldots, V_n$. For a given pair of distinct vertices, $V_i$ and $V_k$, there may or may not be a directed edge from $V_i$ to $V_k$. If there is a directed edge from $V_j$ to $V_k$, we denote the weight of the edge as $e_{jk} > 0$. In addition, we permit a directed edge from a vertex $V_j$ to itself, with an edge weight $e_{jj} > 0$. The topology of the graph can be represented using an $n \times n$ adjacency matrix $P$ defined as follows:

$$P(j, k) = e_{jk}, \text{ if there is a directed edge from } V_j \text{ to } V_k$$

$$= 0, \text{ otherwise.}$$  \hspace{1cm} (1)

We denote the eigenvalues of $P$ as $\lambda_1, \lambda_2, \ldots, \lambda_n$, where without loss of generality (WLOG) we assume that $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. From classical results on non-negative matrices, we immediately see that $P$ will have a real eigenvalue whose magnitude is at least as large in magnitude as any other eigenvalue; let us label this eigenvalue as $\lambda_1$. In the case that the graph is strongly connected, this real eigenvalue in fact has algebraic multiplicity 1 and is strictly dominant (Gallager, 1996). It is worth pointing out that the remaining eigenvalues of $P$ may be complex, and may have algebraic or geometric multiplicities greater than one, and may have associated Jordan blocks of size greater than 1. We use the notation $v_i$ for any right eigenvector associated with the eigenvalue $\lambda_i$. 
In this paper, we study the effect of graph edge-weight increases on the adjacency matrix's eigenvectors, for three different classes of graphs. For each graph class, we consider particular structured topology changes (edge-weight increments). The graph classes and topology modifications that we focus on are representative of engineered modifications of networks, and also permit substantial analysis of eigenvectors (as detailed below). Let us enumerate the particular cases considered here, in order of increasingly general graph classes.

Before presenting these cases, we remark that we denote the $p$th component of a vector $x$ as $x^p$, throughout the remainder of the article.

1. **Symmetric weight increments to undirected (symmetric) graphs.** As a first case, we focus on the class of graphs such that $e_{jk} = e_{kj}$ for any pair of the distinct vertices $V_j$ and $V_k$. We refer to these graphs as undirected or symmetric graphs, and note that the adjacency matrix is symmetric. For undirected graphs, the eigenvalues of $P$ are real and all are in Jordan blocks of size one. Let the eigenvalues of $P$ be in decreasing order of magnitude $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ with associated eigenvectors $v_1, v_2, \ldots, v_n$. The topology change that we examine for the undirected graph case is as follows: for a given pair of distinct vertices $V_j$ and $V_k$, the edge weights $e_{jk}$ and $e_{kj}$ are identically incremented by a certain non-negative quantity, say, $\alpha$, where $\alpha \leq \min(e_{jj}, e_{kk})$. The self-loops on the vertices $V_j$ and $V_k$, or $e_{jj}$ and $e_{kk}$, are reduced by the same amount $\alpha$. That is, we consider modifications that increase the interaction strengths between two vertices at the cost of self-influences or weights. The graph resulting from such a topology change is also an undirected graph. Let us denote the adjacency matrix of the resulting graph by $\hat{P}$. It is easy to see that $\hat{P} = P + \Delta P$, where $\Delta P$ is the following symmetric matrix:

$$
\Delta P(p, q) = \begin{cases} 
-\alpha & \text{for } p = j \text{ and } q = j \\
\alpha & \text{for } p = j \text{ and } q = k \\
\alpha & \text{for } p = k \text{ and } q = j \\
-\alpha & \text{for } p = k \text{ and } q = k \\
0 & \text{otherwise}
\end{cases}
$$

(2) **Balanced weight increments to reversible graphs.** We consider the class of graphs for which (i) the weights of the edges leaving each vertex sum to 1 (i.e., $\sum_k e_{jk} = 1, \forall j$) and (ii) there exists a unique vector $\pi$, such that $\|\pi\|_1 = 1$ and $\pi^T e_{kj} = \pi^T e_{jk}$ for each graph edge. The graphs associated with homogeneous-reversible Markov chains have these properties, and so we refer to these graphs as reversible ones. We note that such reversible graphs also arise in e.g. consensus algorithms for sensor networks (Wan et al., 2012). The adjacency matrix for a reversible graph is a stochastic matrix, for which $\pi$ is a left eigenvector associated with the dominant eigenvalue at unity. In addition to these properties, the adjacency matrix $P$ is also symmetrisable through a diagonal similarity transformation, specifically $A = DP D^{-1}$ is symmetric if $D = \text{diag}(\pi^{0.5})$. (Here the $\text{diag}(x)$ operator on an $q$-component vector $x$ returns a $q \times q$ diagonal matrix which has the components of $x$ along the main diagonal). For this case, we note that the eigenvalues of $P$ are real and non-defective. We denote them as $1 = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, with $v_i$ denoting the right eigenvector associated with eigenvalue $\lambda_i$.

Let us now discuss the topology change we consider in our development. Motivated by both Markov chain and sensor network consensus applications, we consider topology changes that maintain the reversibility structure and leave the left eigenvector associated with the adjacency matrix unchanged. Specifically, the edge weight between the two vertices $V_j$ and $V_k$, or $e_{jk}$, is incremented by an amount $\alpha > 0$, while the self-loop edge weight $e_{jj}$ is reduced by the same amount. Similarly, the edge weight $e_{kj}$ is increased by an amount $\beta > 0$, while the edge weight $e_{kk}$ is reduced by the same amount. Further, the quantities $\alpha$ and $\beta$ are assumed to satisfy:

$$
\frac{\alpha}{\beta} = \frac{\pi^T e_{jj}}{\pi^T e_{kk}}.
$$

Let the adjacency matrix of the resulting graph be $\hat{P}$. It is easy to see that $\hat{P} = P + \Delta P$, where $\Delta P$ is the following matrix:

$$
\Delta P(p, q) = \begin{cases} 
-\alpha & \text{for } p = j \text{ and } q = j \\
\alpha & \text{for } p = j \text{ and } q = k \\
\beta & \text{for } p = k \text{ and } q = j \\
-\beta & \text{for } p = k \text{ and } q = k \\
0 & \text{otherwise}
\end{cases}
$$

(3) **Inhomogeneous structural changes.** The vector $\pi$ is the left eigenvector of $\hat{P}$ associated with the dominant eigenvalue at unity, since
\[ \frac{\beta}{\alpha} = \frac{\pi^i}{\pi^r}. \] Hence, \( \pi \Delta P = 0 \). Further,

\[
\pi^i \hat{P}(j,k) = \pi^i P(j,k) + \pi^i \alpha \\
= \pi^i P(k,j) + \pi^i \beta \\
= \pi^k \hat{P}(k,j)
\] (4)

Therefore, \( \hat{P} \) is also the adjacency matrix of a reversible graph. Let the eigenvalues of the \( \hat{P} \) be \( 1 = \hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_n \), with \( \hat{v}_i \) denoting the right eigenvector associated with eigenvalue \( \hat{\lambda}_i \). We refer to this type of topology change as balanced weight increments to reversible graphs. Our aim is to characterise the subdominant eigenvalues and eigenvectors of \( \hat{P} \) as compared to those of \( P \). We present the results for this case in Section 4.

Let us briefly discuss the relevance of the above problem to the analysis of Markov chains. For a Markov chain, the adjacency matrix serves as the transition matrix of the chain, and the entries of the transition matrix represent conditional probabilities of state transition (Gallager, 1996), i.e., \( P(p,q) \) is the probability of the chain transitioning to state \( q \) given that the current state is \( p \). The vector \( \pi \) specifies the steady-state state-occupancy probabilities. Thus, the balanced topology changes described above can be seen as changing transition probabilities in a way that maintains the steady-state distribution as well as reversibility. Thus, the analysis of this case provides insight into how transition probabilities in a reversible Markov chain can be modified so as to maintain the steady state while shaping dynamical performance.

(3) **Arbitrary increments to weighted graphs.** The previous cases are limited to special graph classes (undirected, reversible), and also consider only very specially structured changes to the topology changes that can be made to the topologies. In this article, we also consider a class of quite general modifications to arbitrary directed graphs. For this case, we focus exclusively on characterising the dominant eigenvector of the modified topology. Specifically, in Section 5, we present three results — two concerned with incrementing weights of the edges departing from multiple vertices and one concerned with weight increments on edges entering into a given vertex. We also present results on designing the graph topology to achieve certain ratios among the dominant eigenvector components. To avoid confusion with cases (1) and (2) above, we defer the mathematical description of arbitrary topology changes to Section 5.

### 3. Symmetric topology changes in an undirected graph

In this section, we develop majorisation results for eigenvector components of adjacency matrices, upon application of symmetric topology changes to undirected graphs, as described in Section 2, case (2). First, we present a preliminary theorem which indicates that when the eigenvectors of the adjacency matrices have a certain special structure, the associated eigenvalues and the eigenvectors themselves are invariant to the defined symmetric changes in the topology (Theorem 1). This straightforward, and well-known, result serves as a starting point for more general spectral analyses, as given in Theorem 2. We include it here for the sake of completeness.

**Theorem 1:** Consider a symmetric weight increment between the vertices \( V_j \) and \( V_k \) in an undirected graph, as described in Section 2, case (2). If \( v_i = v_i \) for some eigenvector \( v_i \) associated with the eigenvalue \( \lambda_i \) of \( P \), then \( v_i \) is also an eigenvector of \( \hat{P} \) with eigenvalue \( \hat{\lambda}_i \). In other words, the eigenvalue \( \hat{\lambda}_i \) and its eigenvector \( \hat{v}_i \) are invariant to the topology change.

**Proof:** The proof follows immediately from the fact that \( \hat{P} v_i = (P + \Delta P) v_i = \lambda_i v_i + \Delta P v_i \). However, we automatically find that \( \Delta P v_i = 0 \), and so the result follows. 

Theorem 1 provides us with a mechanism for changing undirected graph’s structure, in such a manner that particular eigenvectors of its adjacency matrix remain unchanged. The above invariance result may prove applicable for algorithm design in sensor networks, in cases where changing the underlying network topology is a primary tool for improving performance; the result shows how changes can be made that do not affect critical mode shapes, while the remaining dynamics is changed. However, the result is restrictive, in the sense that invariance results hold only when edges are added between vertices whose eigenvector components are identical.

The simple result of Theorem 1 serves as a stepping stone towards a more general characterisation. Specifically, for symmetric topology changes between arbitrary vertices in an undirected graph, we find that the components of the subdominant eigenvector associated with the two ends of the added edge come closer to each other; similar results hold for eigenvectors associated with other eigenvalues, in a certain sequential sense. This result is formalised in the following theorem.

**Theorem 2:** Consider a symmetric weight increment between the vertices \( V_j \) and \( V_k \) in an undirected graph, as described in Section 2, case (2). Then, \( \hat{\lambda}_i = \lambda_i \), with equality only if \( v_i = v_i \). Moreover, \( |\hat{v}_i - v_i| \leq |v_i - v_i| \), with equality only if \( v_i = v_i \).

Furthermore, if \( \lambda_q = \hat{\lambda}_q \) and \( v_q = \hat{v}_q \) \( \forall q \leq p \), where \( p < n \) is some positive integer, then (1) \( \lambda_{p+1} \leq \lambda_{p+1} \),
with equality only if \(v^j_{p+1} = v^k_{p+1}\), and (2) \(|\tilde{\gamma}^{j}_{p+1} - \tilde{\gamma}^{k}_{p+1}| \leq |v^j_{p+1} - v^k_{p+1}|\).

**Proof:** The majorisation of eigenvalues specified in the theorem is akin to existing eigenvalue-majorisation results (Stewart & Sun, 1990), but is included here for the sake of completeness and because the eigenvector majorisation follows thereof. The majorisation of eigenvector components is then proved, by using the Courant–Fischer theorem to arrive at a contradiction.

The topology change is structured in such a way that the adjacency matrices, \(P\) and \(\tilde{P}\) are symmetric. Hence, the eigenvalues (as well as the corresponding eigenvectors) of both the matrices can be found using the Courant–Fischer theorem. In particular, \(\lambda_1\) and \(\tilde{\lambda}_1\) can be written as

\[
\lambda_1 = \max_x (x^T P x) \tag{5}
\]

and

\[
\tilde{\lambda}_1 = \max_y (y^T \tilde{P} y) = \max_y (y^T (P + \Delta P) y) = \max_y (y^T P y - \alpha (y^k - y^j)^2) \tag{6}
\]

where the maxima are taken over vectors of unit length, i.e. \(\|x\|_2 = 1\) and \(\|y\|_2 = 1\). The vectors that achieve the maxima are the eigenvectors associated with the eigenvalues \(\lambda_1\) and \(\tilde{\lambda}_1\). Notice that \(\alpha(y^k - y^j)^2 \geq 0\) for any vector \(y\), with equality only if \(y^j = y^k\). It is therefore straightforward to see that \(\tilde{\lambda}_1 \leq \lambda_1\), with equality only if \(v^j_i = v^k_i\).

To prove the result on the change in eigenvector entries, we again use the Courant–Fischer theorem (Equations (5) and (6)) to arrive at a contradiction. Let us denote the vectors that achieve the maximums as \(x^*\) for Equation (5) and \(y^*\) for Equation (6). Also, assume that \(|(x^*)^k - (x^*)^j| = (y^*)^k - (y^*)^j|\). Then according to Equation (6), we see that

\[
x^*^T P x^* - \alpha(x^k - x^j)^2 \leq y^*^T P y^* - \alpha(y^k - y^j)^2 \tag{7}
\]

Noting that \(x^*^T P x^* \geq y^*^T P y^*\), let \(x^*^T P x^* = y^*^T P y^* + C\) for some \(C \geq 0\). Thus, we find that

\[
\alpha((x^*)^k - (x^*)^j)^2 \geq C + \alpha((y^*)^k - (y^*)^j)^2 \tag{8}
\]

Equation (8) implies that \(|(x^*)^k - (x^*)^j| \geq |(y^*)^k - (y^*)^j|\), which is a contradiction to our assumption. Thus, we have \(|(x^*)^k - (x^*)^j| \geq |(y^*)^k - (y^*)^j|\).

Let us now verify that the inequality on eigenvector components is strict unless the eigenvector components are identical in the unperturbed case. To do so, let us now examine the implication of \(|(x^*)^k - (x^*)^j| = |(y^*)^k - (y^*)^j| \neq 0\) on Equation (6):

\[
y^* = \arg \max_y (y^T P y - \alpha(y^k - y^j)^2) = \arg \max_y (y^T P y - \alpha((x^*)^k - (x^*)^j)^2) = \arg \max_x (x^T P x) \tag{9}
\]

Therefore, if \(|(y^*)^k - (y^*)^j| = |(x^*)^k - (x^*)^j|\), then \(y^* = x^*\), i.e. \(x^*\) is the dominant eigenvector direction for both \(P\) and \(\tilde{P}\), afforded by \(\lambda\) and \(\tilde{\lambda}\). This implies that

\[
\tilde{P} x^* = P x^* + \Delta P x^* = \lambda x^* + \Delta P x^* \tag{10}
\]

Clearly, \(\Delta P x^*\) is not a scaling \(x^*\). Therefore, we can conclude that \(|(y^*)^k - (y^*)^j| < |(x^*)^k - (x^*)^j|\). The result on the eigenvectors \(v_1\) and \(v_1\) thus follows directly from this conclusion.

The Courant–Fischer theorem also permits computation of the eigenvalues \(\lambda_2, \ldots, \lambda_n\). These eigenvalues can be found by optimising the same objective as in Equations (5) and (6), but with the additional constraints that the vector(s) that achieve the maximum(s) should be orthogonal to the subspace spanned by the eigenvectors associated with eigenvalues larger in magnitude. The result on \(v_{p+1}\) and \(\tilde{v}_{p+1}\), when \(\lambda_q = \tilde{\lambda}_q\) and \(v_q = \tilde{v}_q\), \(\forall q \leq p\), follows directly from this fact. \(\square\)

The result of Theorem 2 shows that, for generic undirected graphs (where \(v^j_i \neq v^k_i\)), symmetrically increasing the edge weights between \(V_i\) and \(V_j\) at the expense of the self-loops strictly decreases the dominant eigenvalue as well the difference between the corresponding eigenvector components. This result shows how topology changes impact the eigenvector centrality measure, which captures the relative importance of a vertex in a network as the corresponding dominant graph-adjacency eigenvector component (see Ruhnau, 2000). Our result shows that increasing cross-components interaction between the vertices socialises the importance of the two vertices in the network: specifically, the centrality scores come closer to each other. The result also is informative of participation of network components in dynamics defined by the adjacency matrix; we omit a detailed discussion.

We remark that results similar to Theorem 2 can be developed for the most negative eigenvalue \(\lambda_n\), and the associated eigenvector \(v_n\). However, we have focused our attention on characterising the dominant and subdominant eigenvalues and their eigenvectors, since these spectral properties most often inform characterisations of complex-network dynamics and algorithms.
4. Balanced topology changes in reversible graphs

In Section 3, we limited ourselves to the case where the original matrix and perturbation are symmetric, as described in Section 2, case (2). These eigenvector majorisation results can be extended for balanced topology changes to reversible graphs, as described in Section 2, case (2). This extension is made possible by the observation that a diagonal similarity transformation can be used to equate a reversible graph’s adjacency matrix with a symmetric matrix. The following theorem presents the result for reversible topologies formally.

**Theorem 3:** Consider a reversible graph topology, and assume that a balanced weight increment is performed between the vertices $V_j$ and $V_k$, as described in Section 2, case (2). These eigenvector majorisation results imply that the eigenvalues/eigenvectors before and after the perturbation will have the following properties:

1. If $v_j^i = v_k^i$ for some $i = 1, \ldots, n$, then $\lambda_i = \hat{\lambda}_i$ and $v_i = \hat{v}_i$.
2. If $v_j^i \neq v_k^i$, then $\hat{\lambda}_2 < \lambda_2$ and $|(\hat{v}_j^i - \hat{v}_k^i)| < |(v_j^i - v_k^i)|$.

Moreover, if $\lambda_q = \hat{\lambda}_q$ and $v_q = \hat{v}_q$, $\forall q < p$, where $p < n$ is some positive integer, then $\lambda_{p+1} \leq \lambda_{p+1}$ and $|(\hat{v}_{p+1} - \hat{v}_{p+1})| \leq |(v_{p+1} - v_{p+1})|$.

**Proof:** From the discussion in Section 2, case (2), we know that $\hat{P}$ represents a reversible graph. Furthermore, $\pi$ is an eigenvector associated with the unity eigenvalue of $\hat{P}$. Therefore, there exists a similarity transformation matrix $D = \text{diag}(\pi^{0.5})$ such that $\hat{A} = DPD^{-1}$ and $\hat{A} = D\hat{P}D^{-1}$ are symmetric matrices. Moreover, $\Delta A = \hat{A} - A$ is also symmetric, with the following structure:

$$
\Delta A(p, q) = \begin{cases} 
-\alpha & p = j \text{ and } q = j \\
\sqrt{\alpha\beta} & p = j \text{ and } q = k \\
\sqrt{\alpha\beta} & p = k \text{ and } q = j \\
-\beta & p = k \text{ and } q = k \\
0 & \text{otherwise}
\end{cases}
$$

To prove the multiple statements in the theorem, we follow arguments similar to those used in the proofs of Theorems 1 and 2, while using the fact that if $v_i$ is an eigenvector associated with an eigenvalue $\lambda$ of $P$ (or $\hat{P}$), then $Dv_i$ is an eigenvector associated with an eigenvalue $\lambda$ of $A$ (or $\hat{A}$). The proofs are organised into different cases for ease of presentation:

1. If $v_j^i = v_k^i$ then $\Delta P v_i = 0$. Therefore, $\hat{P}v_i = P v_i = \lambda_i v_i$. The result follows immediately.
2. The eigenvalues of $P$ (respectively, $\hat{P}$) are identical to the eigenvalues of symmetric matrix $A$ (respectively, $\hat{A}$), since the matrices are related by a similarity transform. We can thus use the Courant–Fischer theorem to characterise the eigenvalues and eigenvectors of $\hat{P}$ with respect to those of $P$, in analogy with the proof of Theorem 2. Using the fact that $P$ and $\hat{P}$ are stochastic matrices and noting the form of the similarity transform, we see that the subdominant eigenvalues $\lambda_2$ and $\hat{\lambda}_2$ can be written as

$$
\lambda_2 = \max_{x \perp \frac{1}{\sqrt{n}}} (x^T Ax)
$$

and

$$
\hat{\lambda}_2 = \max_{y \perp \frac{1}{\sqrt{n}}} (y^T \hat{A} y)
$$

where we have used the notation $x \perp \frac{1}{\sqrt{n}}$ to indicate that the maximum is taken over all vectors of unit length that are orthogonal to the $n$-component vector whose entries all equal $\frac{1}{\sqrt{n}}$. The vectors $x$ and $y$ that achieve the maxima, subject to the constraint, are the eigenvectors associated with the eigenvalues $\lambda_2$ and $\hat{\lambda}_2$, respectively. We notice that $(\sqrt{\alpha}y^j - \sqrt{\beta}y^k)^2 \geq 0$ for any vector $y$, with equality only if $\sqrt{\beta}y^k = \sqrt{\alpha}y^j$ (or equivalently $v_j^i = v_k^i$). It therefore follows that $\hat{\lambda}_2 < \lambda_2$ if $v_j^i \neq v_k^i$.

To prove the result on the components of the subdominant eigenvectors, we again use Courant–Fischer theorem (Equations (12) and (13)) to arrive at a contradiction. Let us denote vectors that achieve the maxima (subject to the constraints) as $x^*$ for Equation (12) and $y^*$ for Equation (13). Also, let us assume (to obtain a contradiction) that $|\alpha y^j - \beta y^k| < |\alpha y^j - \beta y^k|$. Then, using $x^*$ in Equation (13) we see that

$$
x^T Ax^* - (\sqrt{\alpha}x^j - \sqrt{\beta}x^k)^2 \leq y^* T A y^* - (\sqrt{\alpha}y^j - \sqrt{\beta}y^k)^2
$$

Noting that $x^T Ax^* \geq y^* T A y^*$, let $x^T Ax^* = y^* T A y^* + C$ for some $C \geq 0$. Thus, we find that

$$
\lambda_2 - (\sqrt{\alpha}x^j - \sqrt{\beta}x^k)^2 \leq \lambda_2 - C - (\sqrt{\alpha}y^j - \sqrt{\beta}y^k)^2
$$

Equation (15) implies that $|\alpha y^j - \beta y^k| \geq |\alpha y^j - \beta y^k|$, which is a contradiction to our assumption.
In the case that \(|\sqrt{\alpha}(x^*) - \sqrt{\beta}(x^*)| = |\sqrt{\alpha}(y^*) - \sqrt{\beta}(y^*)| \neq 0\), one can argue that
\[
y^* = \arg \max_{(y \perp \lambda^* y x^* - \sqrt{\beta} x^*)} (y^* y x^* - \sqrt{\beta} x^* y^*)
\]
which leads to the conclusion that \(y^* = x^*\).

Clearly, \(\Delta A x^*\) is not proportional to \(x^*\) and therefore, \(x^*\) cannot be an eigenvector of \(A\). Thus, we see that
\[
|\sqrt{\alpha}(y^*) - \sqrt{\beta}(y^*)| < |\sqrt{\alpha}(x^*) - \sqrt{\beta}(x^*)|.
\]

The result on the eigenvectors \(v_2\) and \(\hat{v}_2\) follows from this contradiction, from the fact that \(v_2 = D^{-1} x^*\) and \(\hat{v}_2 = D^{-1} y^*\), and from the assumed relationship between \(P(j, k)\) and \(P(k, j)\).

The results on \(v_2 + 1\) and \(\hat{v}_{p+1}\), under the condition that \(\hat{v}_p = v_p\) and \(\lambda_p = \lambda_p\) \(\forall p < n\), can be proved using a very similar argument.

The result of Theorem 3 shows that, for generic reversible graph topologies under balanced weight increments, the dominant eigenvalue as well as the difference between eigenvector components strictly decreases in magnitude. As we pointed out while formulating the problem in Section 2, the result provides a mechanism for changing state-transition probabilities of a reversible Markov chain to design the steady-state state-occupancy probabilities (Gallager, 1996).

5. Arbitrary weight increments in a directed graph

The results developed in Sections 3 and 4 consider specific changes to undirected or reversible graph topologies. A characterisation of the change in the eigenvalues and/or eigenvectors for general topology modifications in arbitrary graphs has wider applications. In this section, we present some characterisations of the dominant eigenvector, for more general topology changes in arbitrary directed and weighted graphs.

To avoid confusion between these results and those developed in Sections 3 and 4, we use slightly different (and simplified) notations here. For the remainder of the paper, we consider a directed and weighted graph of \(n\) vertices labelled \(V_1, V_2, \ldots, V_n\), with an adjacency matrix denoted as \(P\). We use \(\tilde{P}\) to denote the adjacency matrix of a graph resulting from a change in the topology. We assume that both \(P\) and \(\tilde{P}\) capture strongly connected graphs, in which case they both have positive real eigenvalues of algebraic multiplicity 1 that are larger in magnitude than all other eigenvalues, i.e. that are dominant. The dominant eigenvectors of \(P\) and \(\tilde{P}\) are denoted as \(v\) and \(\tilde{v}\) respectively, while the dominant eigenvalues are denoted as \(\lambda\) and \(\tilde{\lambda}\) respectively. For a vector \(x\), the notation \(x_{p, q}\) specifies a \((q - p + 1)\)-component subvector, which contains the correspondingly indexed components of the \(x\) (i.e. the components between the \(p\)th and \(q\)th components, inclusive).

The outline of this section is as follows. In Theorem 4, we present a characterisation of the dominant eigenvector when weights of edges emerging from a single vertex are arbitrarily increased. Such a topology change corresponds to arbitrarily incrementing a given row, say (WLOG), the first row, of \(P\). In Theorem 5, we consider a generalisation of Theorem 4 to the case where we arbitrarily increment the first \(m < n\) rows of \(P\). This corresponds to arbitrarily incrementing the weights of edges emerging from the \(m\) given vertices. In Theorem 6, we present results on the arbitrary weight increments to edges directed into a given vertex. This corresponds to arbitrarily incrementing a single column, say, the first column, WLOG, of the graph adjacency matrix.

In addition to these results, we also present a design result, that shows how self-loops of a directed and weighted graph can be modified to achieve certain ratios among the dominant eigenvector components. This result is presented in Theorem 7, and directly informs the multi-row incrementation result.

Let us present the first result, regarding arbitrary changes to the outgoing edges from a single vertex in a directed graph.

**Theorem 4:** Consider arbitrarily increasing the weights of the edges emerging from the vertex \(V_1\). Then, \(\frac{\hat{v}_1}{v_1} > \frac{\hat{v}_1}{v_1}\) and for \(j = 2, \ldots, n\). We thus recover that, when the eigenvector is normalised to unit length, \(\hat{v}_1 > v_1\).

Furthermore, it can be shown that \(\hat{v}_1 > \alpha v_1\), where \(\alpha > 1\) is a quantity that depends on \(\Delta = \tilde{\lambda} - \lambda\) and the dominant eigenvalue of the principal minor of \(P\) obtained by deleting its first row and column.

**Proof:** It is well known, see, e.g. Berman and Plemmons (1979), that incrementing entries of an irreducible non-negative matrix strictly increases its dominant eigenvalue, so in our case \(\tilde{\lambda} = \lambda + \Delta\) for some \(\Delta > 0\). Also, from the Perron–Frobenius theorem (Gallager, 1996), the dominant eigenvectors \(v\) and \(\tilde{v}\) are entry-wise positive.

To prove the majorisation on the eigenvector components we consider a particular scaling of the dominant eigenvectors. Specifically, we scale the dominant eigenvector after perturbation so that the first entry is the same as for the original eigenvector. That is, we assume the dominant eigenvector has the form \(\hat{v} = [v_1, v_2 + q]\), is scaled so that \(v_1 = \tilde{v}_1\) and \(q\) is some \((n - 1)\)-component vector.

We shall prove that the vector \(q\) is (element-wise) strictly negative. To do so, let us simply work from the eigenvector equation \(\tilde{P}\hat{v} = \tilde{\lambda}\hat{v}\). Let us consider the last \(n - 1\) equations
in the system of equations. Specifically, substituting for \( \hat{\lambda} \) and \( \hat{v} \), and using the eigenvector equation \( P\hat{v} = \lambda \hat{v} \), we find that

\[
P_{2,n,2,n} q = \hat{\lambda} q + \Delta v_{2,n}
\]

where \( P_{2,n,2,n} = \begin{bmatrix} p_{21} & \cdots & p_{2n} \\ \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix} \). Rearranging, we obtain

\[
(\hat{\lambda} I - P_{2,n,2,n}) q = -\Delta v_{2,n}.
\]

From classical results, we have that the dominant eigenvalue of the non-negative matrix \( P_{2,n,2,n} \) is less than or equal to \( \lambda \), and hence strictly less than \( \hat{\lambda} \) (Stewart & Sun, 1990). Thus, \( \hat{\lambda} I - P_{2,n,2,n} \) is a non-singular M-matrix (Berman & Plemmons, 1979). It is thus automatic that its inverse is element-wise non-negative. In fact, noting that each irreducible submatrix of \( P_{2,n,2,n} \) has a strictly positive inverse, we obtain that \( (\hat{\lambda} I - P_{2,n,2,n})^{-1} \) has at least one strictly positive entry on each row. Noting that \( v_{2,n} \) is entry-wise strictly positive, we thus find that \( q \) is entry-wise strictly negative. The majorisations in the theorem statement follow immediately.

Before we present a proof about our claim about the factor \( \alpha \), let us introduce two temporary notations:

- \( \lambda_{\text{dom}}(P_{2,n,2,n}) \) is the dominant eigenvalue of \( P_{2,n,2,n} \)
- \( \lambda_{\text{diff}} = \hat{\lambda} - \lambda_{\text{dom}}(P_{2,n,2,n}) \)

Notice that from the above equation (as well as our assumption on connectivity of graph defined by \( P \) and \( \hat{P} \)) we have

\[
q = -\Delta (\hat{\lambda} I - P_{2,n,2,n})^{-1} v_{2,n} \\
\geq -\frac{\Delta}{\hat{\lambda} - \lambda_{\text{dom}}(P_{2,n,2,n})} v_{2,n}.
\]

The second statement follows from the fact that the smallest eigenvalue of the non-singular M-matrix \( \hat{\lambda} I - P_{2,n,2,n} \) is \( \lambda \) and \( \lambda_{\text{dom}}(P_{2,n,2,n}) \). We stress here that the statement is an element-wise comparison. Since \( q \) is element-wise strictly negative, we see that

\[
|q| \leq \frac{\Delta}{\hat{\lambda} - \lambda - \lambda_{\text{dom}}(P_{2,n,2,n})} v_{2,n} \\
\leq \frac{\Delta}{\hat{\lambda} + \lambda_{\text{diff}}} v_{2,n}.
\]

Therefore, we see that the change in the eigenvector components can be bounded in terms of \( \Delta \) and \( \lambda_{\text{diff}} \). This concludes our proof.

The above theorem shows that when a particular row of a positive matrix is arbitrarily incremented, the associated component of the dominant eigenvector increases relative to the other components. Although this paper is motivated by the graph topology change perspective, these results can be used in other application areas. We refer readers to Roy et al. (2008) for a conceptual discussion on how these results can be used in velocity control problems for autonomous-vehicle-coordination applications and virus-spreading control applications. In Roy et al. (2008), we also presented some corollaries to Theorem 4 from a linear algebra point of view.

Equation (19) provides an interesting insight. It aligns with the intuition that in the limit of small \( \Delta \), there is no change in the dominant eigenvector components. In the limit of large \( \Delta \) (which is the case if weights of the edges emerging from the vertex \( V_1 \) are increased by arbitrarily large quantities), the eigenvector direction becomes arbitrarily close the first standard basis vector. Similarly, the quantity \( \lambda_{\text{diff}} \) is an inherent property of the graph defined by \( P \), and does not depend on the specifics of how the weights of the edges emerging from the vertex \( V_1 \) are increased. It is the amount by which the dominant eigenvalue would decrease if the vertex \( V_1 \) (along with all the edges associated with it) is deleted from the graph and captures a degree of importance of \( V_1 \).

We next present a characterisation of the dominant eigenvector of adjacency matrix, when weights of the edges emerging from the \( m \) given vertices are incrementally arbitrarily increased. We note here that eigen structure of the adjacency matrix is graph invariant within a permutation, i.e. the order of the \( m \) vertices is not important; thus we focus on incrementing the first \( m \) rows WLOG.

**Theorem 5:** Consider arbitrarily increasing the weights of the edges emerging from the vertices \( V_1, V_2, \ldots, V_m \).

Then \( \exists j \in \{1, 2, \ldots, m\} \) such that \( \frac{\hat{v}_j}{\hat{v}_i} > \frac{v_j}{v_i} \) for \( j = m + 1, \ldots, n \). We thus recover that when the eigenvector is normalised to unit length, \( \hat{v}_i > v_i \).

**Proof:** Incrementing rows strictly increases the dominant eigenvalue for a non-negative matrix associated with a connected graph (Stewart & Sun, 1990); thus, \( \hat{\lambda} = \lambda + \Delta \) for some \( \Delta > 0 \).

To prove the majorisation on the eigenvector components, we consider a particular scaling of the dominant eigenvectors. Specifically, let the dominant eigenvectors \( \hat{v} = \begin{bmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{m1} & \cdots & v_{mn} \end{bmatrix} \), be scaled so that \( q_1 \) is an \( m \) component non-positive vector with \( q_{1i} = 0 \) for some \( i \in \{1, \ldots, m\} \).

Notice that this scaling can be achieved by scaling down the original eigenvector \( \hat{v} \) after perturbation by the largest among the ratios \( \frac{\hat{v}_i}{v_i} \) for \( i = 1, \ldots, m \).

We shall prove that \( q_2 \) is element-wise strictly negative for the assumed scaling. To prove that, let us consider the last \( n - m \) equations of the eigenvector equation \( \hat{P}\hat{v} = \hat{\lambda}\hat{v} \). For notational convenience, we partition the matrix \( P \) (and similarly, \( \hat{P} \)) as

\[
P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}
\]
such that $P_{11}$ is an $m \times m$ matrix and the dimensions of the other partitions are consistent. Substituting $\Delta$ and $\hat{\lambda}$ into the eigenvector equation $\hat{\lambda} \hat{v} = \hat{\lambda}^* \hat{v}$, and also noting the assumed form for $\hat{v}$, we obtain

$$ (\hat{\lambda} I - P_{22}) q_2 = P_{21} q_1 - \Delta v_{(m+1):n} $$

It is easy to see that $(\hat{\lambda} I - P_{22})$ is an M-matrix. Using the fact that $q_1$ is entry-wise non-positive and that the inverse of an M-matrix is a non-negative matrix, we see that (for the assumed scaling of the $\hat{\lambda}$) the vector $q_2$ is entry-wise strictly negative. The majorisation result in the theorem statement follows immediately. □

The result presented in Theorem 5 gives us insight into how changes to the topology of an arbitrary directed graph would affect the eigenstructure of the adjacency matrix. We now turn our attention to the case where weights incoming to the first vertex also increase the edge weights to the $i$th vertex also increases the $i$th vertex also increases the eigenvector components of the eigenvector. We present this result in the following theorem.

**Theorem 6:** Consider arbitrarily increasing edge weights incoming to the first vertex (WLOG). Let $E = \hat{P} - P$ and $\Delta = \hat{\lambda} - \lambda$. If $(\Delta I - E)$ is non-negative, then $\frac{\hat{v}_i}{v_i} > \frac{\hat{v}_j}{v_j}$ and for $j = 2, \ldots, n$. We thus recover that, when the eigenvector is normalised to unit length, $\hat{v}_1 > v_1$.

**Proof:** It is easy to see that $\Delta > 0$ (Stewart & Sun, 1990). Moreover, from the Perron–Frobenius theorem for non-negative matrices, the dominant eigenvectors are entry-wise non-negative (Gallager, 1996). We consider the scaling of the eigenvector $\hat{v}$ for which $\hat{v} = \begin{bmatrix} v_1 \\ v_2 + q \end{bmatrix}$ where $q$ is a $n - 1$ length vector. We wish to find conditions for which $q$ is element-wise strictly negative, using perturbation results.

In order to find such conditions, we study the eigenvector equation $\hat{P} \hat{v} = \hat{\lambda} \hat{v}$. Substituting for $\hat{\lambda}$ and $\hat{v}$, we arrive at the following equation:

$$ (\hat{\lambda} I - P) \begin{bmatrix} 0 \\ q \end{bmatrix} = (E - \Delta I)v $$

If $(\Delta I - E)$ is a non-negative matrix, then it can be seen that $q$ has strictly negative entries. □

In order to find conditions for which $(\Delta I - E)$ is a non-negative matrix, we can use the various perturbation results for general matrices such as the Ostrowski–Elsner theorem (Stewart & Sun, 1990). We would also like to point out that our results, presented in Theorems 5 and 6, allow for design of new edges that modify the eigenvectors in a specified way.

In the results developed thus far, we have characterised the dominant eigenvector components of the graph-adjacency matrix when the graph topology is arbitrarily changed. Results on how topology changes can be designed to specifically meet a desired dominant eigenvector structure are also useful. However, developing such results is non-trivial. As a step in that direction, we next present a result on the existence of a topology change that can achieve certain design specifications.

Specifically, in Theorem 7, we present a design result on incrementing the multiple diagonal entries to achieve certain ratios among the eigenvector components. The design result is motivated by decentralised controller design problems, wherein the dependence of dominant eigenvector components on multiple diagonal entries’ incrementation are important (Roy & Saberi, 2007; Wan et al., 2008). Let us proceed to the design result.

**Theorem 7:** Consider a graph with an adjacency matrix $P$, with dominant eigenvector $v$ associated with the dominant eigenvalue $\lambda$. Assume that we can increase the weights on the self-loops of first $m < n$ vertices by amounts $k_1, \ldots, k_m$, i.e. $\forall i \in (1, \ldots, m)$, i.e. $P_{ii}$ can be increased by $k_i$. Let the adjacency matrix of the graph resulting from incrementing weight of the self-loops be $\hat{P}$ with a dominant eigenvector $\hat{v}$ associated with the dominant eigenvalue $\hat{\lambda}$.

Then for each $k_1 > 0$, we can find $k_2 > 0, \ldots, k_m > 0$ so that

1. $\frac{\hat{v}_i}{i} = \frac{v_i}{i}$ for $i = 1, \ldots, m$ and $j = 1, \ldots, m$.
2. $\frac{\hat{v}_i}{i} > \frac{v_i}{i}$ for $i = 1, \ldots, m$ and $j = m + 1, \ldots, n$.

Furthermore, $k_2, \ldots, k_m$ increase monotonically with $k_1$.

**Proof:** The design result can be proved using a generalisation of Theorem 5 as follows. We first assume that there exist a set of values $k_1, k_2, \ldots, k_m$ such that

$$ \frac{\hat{v}_i}{i} = \frac{v_i}{i} \text{ for } i = 1, \ldots, m \text{ and } j = 1, \ldots, m $$

(20)

We show that the second majorisation result presented in the theorem follows from this assumption. Then, we show that there indeed is a choice for $k_1, k_2, \ldots, k_m$ such that (20) holds, we see that there exists a scaling of the dominant eigenvector $\hat{v}$ such that

$$ \hat{v} = \begin{bmatrix} v_{1,m} \\ v_{m+1,n} + q_2 \end{bmatrix} $$

(21)

Incrementing row entries strictly increases the dominant eigenvalue, i.e. $\hat{\lambda} = \lambda + \Delta$ (Stewart & Sun, 1990). Let us now consider the last $n - m$ equations of the eigenvector
equation \( \hat{P}\hat{\nu} = \lambda \hat{\nu} \). For notational convenience, we partition the matrix \( P \) (and similarly, \( \hat{P} \)) as

\[
P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}
\]
such that \( P_{11} \) is an \( m \times m \) matrix and the dimensions of the other partitions are consistent. Substituting \( \Delta \) and \( \hat{\nu} \) into the eigenvector equation \( \hat{P}\hat{\nu} = \lambda \hat{\nu} \), and also noting the assumed form for \( \hat{\nu} \), we obtain

\[
(\hat{\lambda}I - P_{22})q_2 = -\Delta v_{(m+1)n}
\]

(22)

Notice that \( (\hat{\lambda}I - P_{22}) \) forms a non-singular M-matrix. Using properties of M-matrices, positivity of \( \Delta \), and entries of \( v_{m+1,n} \), we see that \( q_2 \) is entry-wise strictly negative. The majorisation result presented in the second statement follows directly from this fact.

We next need to show that there exist a set of strictly positive weight increments, \( k_1, k_2, \ldots, k_m \), for the self-loops at first \( m \) vertices which results in Equation (20). This can be done by considering the first \( m \) equations of the eigenvector equation \( \hat{P}\hat{\nu} = \lambda \hat{\nu} \), where \( \hat{\nu} \) is given by Equation (21). Specifically for \( K = \text{diag}(k_i) \) we see that

\[
(P_{11} + K)v_{1,m} + P_{12}(v_{m+1,n} + q_2) = \hat{\nu}_{1,m}
\]

\[
Kv_{1,m} = \hat{\nu}_{1,m} - P_{11}v_{1,m} - P_{12}(v_{m+1,n} + q_2)
\]

(23)

Let us now create an \( m \times m \) diagonal matrix \( V \) with components \( v_1, v_2, \ldots, v_m \) of the eigenvector \( \hat{\nu} \). Thus, we can rephrase the equations in (23) as

\[
Vk = \Delta v_{1,m} - P_{12}q_2
\]

(24)

where \( k \) is an \( m \) length vector of \( k_1, k_2, \ldots, k_m \). It is easy to see that Equation (24) can always be solved explicitly for \( k \) for any choice of \( \Delta \). Substituting \( q_2 \) from Equation (22) in Equation (24), we see that \( \forall i = 1, \ldots, m \) the component \( k_i \) has the form

\[
k_i = \Delta + \alpha_i
\]

(25)

where \( \alpha_i = \frac{P_{12}\hat{\nu}_{1,m} - P_{12}}{\hat{\nu}_{m+1,n}} \geq 0 \). Thus, components of the vector \( k \) have a monotonic dependence on \( \Delta \) and increase unboundedly as \( \Delta \) increases. Solving the equations for every possible value of \( \Delta > 0 \), we obtain a design result for every choice of \( \Delta \). Therefore, we can create a mapping between every possible \( k_i > 0 \) and the corresponding \( \Delta \) that achieves the design constraints on the dominant eigenvector. Thus, we have a design result of the every choice of \( k_1 > 0 \).

Finally, to complete our proof, we need to show the monotonic dependence of \( k_2, \ldots, k_m \) on \( k_1 \). To do so, we note that the components of \( k \) increase monotonically with \( \Delta \), as seen from Equation (25). Hence, every component of \( k \) increases monotonically with respect to \( k_1 \).

The design result is thus proved. \( \square \)

An interpretation of the design result presented in Theorem 7 is that the weights of self-loops can be designed so that corresponding \( m \) components of the dominant eigenvector remain unchanged to within a constant scaling, while the ratio of these \( m \) components to the other components strictly increases. This result is very important from a design perspective. For example, it allows us to modify the local structure of a discrete-time linear autonomous system defined on graph topology or otherwise, to meet various performance criteria. Similarly, the result can be used in mathematical epidemiology to understand the impact of and design various virus-spread mitigation initiatives such as quarantine procedures and vaccination programmes (Wan et al., 2008, 2012).

We note that the result in Theorem 7 shows that there exists a design that achieves the design constraints on the dominant eigenvector of the adjacency matrix. The theorem does not provide closed-form expressions for such topology changes. However, the design can be obtained rather easily using a digital computer, since it only requires search over one parameter (\( \Delta \)).

We conclude our exposition with an illustrative example related to opinion/spread dynamics where the design result may be useful. We focus on a conceptual discussion rather than a numerical example, with the aim of giving a broader perspective on the applications of the developed majorisation.

5.1 Conceptual example

We consider a toy network model for spread (e.g. opinions, fads, ideas, or viruses) among \( n \) agents in a network. In this network model, we track the penetration of the infection for each agent (i.e. the probability or prevalence of infection). The penetration level evolves through interactions with neighbouring agents in the network. In particular, each agent’s evolution depends on two quantities: a set of pre-defined interaction factors that capture the influence of one agent on another (possibly itself), and a desirable stubbornness factor. Let us provide some further technical details of this toy network model.

Let the infection’s penetration at agent \( i \) be captured in a scalar state \( x_i(k) \) at times \( k = 0, 1, \ldots, n \). The infection spreads through the network via interactions with other agents in the networks. In particular, we consider a linearised spread model where the \( \lambda_j \geq 0 \) term captures the influence of agent \( j \) on agent \( i \)’s evolution, while \( \alpha_i \geq 0 \) captures the stubbornness of agent \( i \) (the strength of the agent’s self-influence in the spread). We refer the reader to, e.g. Roy, Xue, and Das (2012) and Wan et al. (2008) for a detailed formulation of spread models of this sort. Assem-
blinding the state evolutions of all the agents in a matrix–vector form, we get
\[
x[k + 1] = (D + P^T)x[k]
\]
(26)
where \(x[k] = [x_1 \ x_2 \ldots \ x_n]^T\), \(P = [p_{ij}]\), \(D = \text{diag}(\alpha_i)\). We note that \(P\) is an adjacency matrix of underlying graph topology that defines the spread.

Asymptotically, the spread dynamics is predominantly governed by the dominant eigenvectors (both left and right) of \(D + P^T\). Specifically, let \(v\) and \(w^T\) be the unit-length right and left eigenvectors associated with the dominant eigenvalue \(\lambda\) of \(P\). For the case where \(D = 0\), i.e. the agents are not stubborn, we can see that, for large \(k\), we have
\[
x[k] \frac{1}{\lambda^k} \to wv^T x[0].
\]

Therefore, the ratios between the right eigenvector components decide the relative contribution of each agent’s initial penetration towards the asymptotically dominant direction of the spread dynamics: infections that begin at agents with large eigenvector components are multiplied compared to those that begin at other agents. From another viewpoint, the higher the component \(v_i\) for agent \(i\), the higher the agent’s contribution to the spread. Meanwhile, the ratios among the components of \(w^T\) give the relative magnitude of the penetration at each agent. The higher the component \(w_i^T\) is for agent \(i\), the more pronounced is the penetration \(x_i[k]\).

Let us assume that we want to increase (or decrease) the initial contribution towards the dominant spread direction of a subset of agents compared to the rest of the network, while maintaining their relative contributions within the subset in the interest of fairness (Vijayshankar & Roy, 2012). The result in Theorem 7 shows that such a design can be achieved by increasing these agents’ stubbornness factors \(\alpha_i\). While we do not provide an analytical expression for \(\alpha_i\), our proof for the theorem indicates a numerical approach for finding them. Thus, it is possible to increase or decrease the influence of a group of agents commensurately (or for that matter in any desired way), through local modifications/controls at those agents. This is a useful observation, if limited resources need to be placed to mitigate a spread, or, alternatively, to propagate an important opinion.

It is worth noting that designing the contributions of the agents to the spread also serves to modify the relative penetration at these components. To see this, notice that \(w\) is the right eigenvector of \(P^T\). By Theorem 5, for at least one of the agents with increased stubbornness, the corresponding component in \(w^T\) strictly increases. In other words, the agent also becomes more susceptible to spread. Thus, increasing the relative influence of a subset of agents implicates a trade-off in spread penetration at those locations.

Although we have focused on influence-shaping in opinion/infection spread, similar models and problems arise in other applications of mathematical epidemiology (Roy et al., 2008, 2012; Wan et al., 2008), and in such other domains as integration of renewable generation in an electric power grid.

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**References**


