A pre- + post- + feedforward compensator design for zero placement

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We demonstrate the design of a pre- + post- + feedforward compensator that places the transmission zeros of a stabilisable and detectable multi-input multi-output linear-time-invariant plant at arbitrary locations.

**Keywords:** invariant zeros; special coordinate basis; compensator architecture

1. Introduction and problem formulation

Relocation of a system’s finite invariant zeros using feedforward controller architecture is of importance in several control applications (e.g. Lozano-Leal 1989; Ortega 1990; Bayard and Boussalis 1993; Bayard 1994; Ortega, Bartolini, and Ferrara 1994), including adaptive control and stable-plant-inversion-based design. In particular, lifting techniques – which parallelise and combine plant inputs and outputs to achieve zero relocation and annihilation – have been developed for several plant models. While researchers have developed a range of lifting techniques, a systematic methodology for placing invariant zeros at desired locations is not known, and in fact the literature makes clear the complexity of the zero-relocation problem (Bayard and Boussalis 1993; Ortega et al. 1994). In this brief communique, we develop a systematic methodology for relocating the finite invariant zeros of a continuous-time multiple input multiple output (MIMO) linear time invariant (LTI) plant using the time-invariant feedforward control architecture shown in Figures 1 and 2. Our zero-relocation methodology exploits the special coordinate basis (SCB) for linear systems (Sannuti and Saberi 1987; Saberi, Chen, and Sannuti 1993).

Precisely, let us consider an arbitrary stabilisable and detectable linear-time-invariant plant:

\[
\dot{x} = Ax + Bu, \\
y = Cx, \tag{1}
\]

where \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, \) and \( x \in \mathbb{R}^{n}, u \in \mathbb{R}^{m} \) and \( y \in \mathbb{R}^{p} \) are the state, input, and output, respectively. We denote the plant in Equation (1) by \( \Sigma_{P} \).

We will demonstrate that precompensation plus postcompensation plus feedforward compensation can be used to place the transmission zeros of the plant at will. That is, we will design compensators \( \Sigma_{pre}, \Sigma_{post} \) and \( \Sigma_{ff} \) such that \( \Sigma = \Sigma_{pre}\Sigma_{P}\Sigma_{post} + \Sigma_{ff} \) (the cascade of \( \Sigma_{P} \) with \( \Sigma_{pre} \) and \( \Sigma_{post} \) all in parallel with \( \Sigma_{ff} \)) is stabilisable and detectable, and has transmission zeros at a set of desired locations. The pre- + post- + feedforward compensator is illustrated in Figure 1.

2. Main result

In this section, we elucidate our zero-relocation design that combines the common approach of squaring down and rank uniformisation with a specific feedforward controller design.

Before presenting the main theorem on the complete compensator design, let us quote two classical lemmas that together demonstrate that a LTI plant \( \Sigma_{P} \) can be made square invertible and uniform rank by adding dynamic compensation. Lemma 1 considers design for square invertibility; see Theorems 3.1 and 3.2 in Saberi and Sannuti (1988) for the full derivation of the design.

**Lemma 1 (Squaring down):** Consider the stabilisable and detectable LTI plant \( \Sigma_{P} \) (Equation (1)). Proper pre- and post-compensators can be designed, so that the resulting plant is (1) square invertible and (2) stabilisable and detectable. The compensated system has the same number of inputs as outputs, and this number is equal to the normal rank of the original plant. The compensators induce an additional set of invariant zeros in the squared-down system, whose locations can be designed.

The algorithm for designing the squaring-down precompensator can be found in Saberi and Sannuti (1988). Upon squaring down, the finite-invariant-zero structure is expanded by \( z \) open left half-plane (OLHP) transmission zeros, where \( z \) is the order of the
Lemma 2 considers adding further precompensation to make the plant uniform rank. Specifically, for a non-uniform rank system (one which has different lengths of infinite-zero chains), the shorter chains can be extended by adding integrators at the input side, and hence the system can be made uniform rank without altering the finite-invariant-zero dynamics. Please see Proposition 4 in Saberi, Kokotovic, and Sussmann (1990) for the derivation.

Lemma 2 (Rank uniformisation): Consider a stabilisable and detectable square invertible plant. A proper precompensator can be designed so that the resulting system is stabilisable and detectable, uniform rank, and its invariant zeros are the same as those of the original plant.

The algorithm for designing the rank-uniformising pre-compensator is given in Saberi, Kokotovic, and Sussmann (1990). Now we are ready to state the main result of this article on zero relocation.

Theorem 1: Consider the stabilisable and detectable LTI plant \( \Sigma_p \) (Equation (1)). A pre- + post- + feedforward compensator of the form shown in Figure 1 can be designed, so that the compensated plant remains stabilisable/detectable, and is (1) square invertible, (2) uniform rank, and (3) minimum phase. In fact, the \( m \) transmission zeros of the compensated plant can be placed at any set of locations \( x_1, \ldots, x_m \) that are closed under conjugation. Furthermore, \( \Sigma_{\text{pre}} \) (which is of dimension \( r \times m \), where \( r \) is the normal rank of the system), \( \Sigma_{\text{post}} \) (which is of dimension \( p \times r \)) and \( \Sigma_{\text{ff}} \) (which is of dimension \( r \times r \)) are all proper.

Proof 1: We shall give an explicit construction of the controller, by designing the compensator blocks around the plant, as shown in Figure 2.

The two lemmas demonstrate the design to make the LTI plant \( \Sigma_p \) square invertible and uniform rank while maintaining stabilisability/detectability by adding dynamic compensation (blocks S.D.1 and S.D.2 for achieving square invertibility and R.U. for rank uniformisation). Henceforth in this proof, we only consider zero relocation of a square invertible and uniform-rank system.

Let us denote the input of the uniform-rank square invertible system as \( u \in \mathbb{R}^r \), the output as \( y \in \mathbb{R}^r \), and the relative degree as \( q \). The system can be written in the (SCB) as (see Saberi et al. (1993) and Sannuti and Saberi (1987) for details):

\[
\begin{align*}
\dot{x}_a &= A_a x_a + A_1 x_1, \\
\dot{x}_1 &= E_a x_a + L(x_1, \ldots, x_q) + CA^{q-1}Bu, \\
\dot{x}_q &= y = x_1,
\end{align*}
\]

where \( A_a \in \mathbb{R}^{n_a \times n_a} \), \( E_a \in \mathbb{R}^{r \times n_a} \) and \( L( ) \) denotes a linear function of the elements in \( ( ) \). Here, the triple \((E_a, A_a, A_1)\) specifies the zero dynamics of the system, and the system zeros are the eigenvalues of \( A_a \).

Next, let us use postcompensation to obtain the output-derivative \( x_q \) from the output \( y \), using blocks D.E. and S.F. in Figure 2. To obtain \( x_q \) from \( y \), we should use an estimator block with transfer function \( s^{q-1}I \), however we require a proper LTI compensator. As an alternative, let us use a high-gain estimator, for instance one with transfer function \( \frac{1}{(s+\epsilon)^{q-1}}I \) with small \( \epsilon \), together with a smoothing filter with transfer \( \frac{1}{(s+\epsilon)^{q-1}}I \) on the feedforward path. We notice that this scheme is equivalent to using a pure derivative estimator \( s^{q-1}I \) before the addition of the feedforward signal, together with a filter with transfer function \( \frac{1}{(s+\epsilon)^{q-1}} \) after the addition (see Figure 3). Thus, the use of the high-gain estimator only serves to introduce a set of poles that are far in the OLHP; we can thus continue the analysis from here on assuming use of the pure derivative compensator. We notice that this compensator serves to cancel the infinite zeros of the plant, and does not introduce any new finite-invariant zeros.

Upon compensation, we can view the dynamics as being the same as the above one, but with output \( \tilde{y} = \tilde{x}_q \). Upon reformulation, a portion of the infinite-zero chains (of length \( q-1 \)) is attached to the previous zero dynamics at its input side and hence, clearly, the new system is uniform rank-1 and remains square invertible, stabilisable and detectable. The SCB of the new system is:

\[
\begin{align*}
\dot{x}_a &= \hat{A}_a \hat{x}_a + \hat{A}_1 \hat{x}_1, \\
\dot{\hat{x}}_1 &= \hat{E}_a \hat{x}_a + \hat{E}_1 \hat{x}_1 + \hat{C}Bu, \\
\tilde{y} &= \hat{x}_1,
\end{align*}
\]

where \( \hat{x} = [x_a, x_1, \ldots, x_{q-1}]^T \), \( \hat{x}_1 = x_q \), and \( \hat{A}_a, \hat{A}_1, \hat{E}_a, \hat{E}_1 \) and \( \hat{C} \) can be obtained from Equation (2) directly. We see that the zeros of the new system contain all the previous zeros plus a number \( q-1 \) of zeros at the origin. Moreover, the zero dynamics \((\hat{A}_a, \hat{A}_1, \hat{E}_a, \hat{E}_1)\) is stabilisable and detectable. This is because the stabilisability and detectability of the reconstructed system in Equation (3) implies that the
S.F. is a smoothing filter with transfer function \( \frac{1}{1+(s + \epsilon)^q} \), where \( \epsilon \) is small and \( q \) is the relative degree of the rank-uniformised plant. S.F. is a smoothing filter after the feedforward addition.

Transmission zeros are controllable and observable modes of the zero dynamics (Sannuti and Saberi 1987) and the decoupling zeros are in OLHP.

Let us construct the feedforward compensator as

\[
\begin{align*}
\dot{x}_c &= A_c x_c + B_c y, \\
y_c &= C_c x_c, \\
u &= G y_c + F y, \\
\tilde{y} &= \tilde{y} - y_c,
\end{align*}
\]

where \( \tilde{y} \) is the new system output, \( G = (\hat{C}B)^{-1} C_c A_c \), and \( F = (\hat{C}B)^{-1}(C_c B_c - I) \). Notice that this feedforward compensator is shown in the blocks labelled ‘static map’ and ‘zero relocation’, as well as in the summation at the output in Figure 2. Let us show that the system with this precompensator can be designed to be minimum phase (and in fact to have transmission zeros at arbitrary locations). To do so, let us find the zeros of the system with this precompensator. By taking \( \tilde{y} = 0 \), and \( \tilde{y} = 0 \), we see that the zero dynamics is

\[
\begin{bmatrix}
\dot{x}_a \\
\dot{x}_c
\end{bmatrix} =
\begin{bmatrix}
\hat{A}_d & \hat{A}_1 C_c \\
B_c E_c & A_c + B_c E_1 C_c
\end{bmatrix}
\begin{bmatrix}
x_a \\
x_c
\end{bmatrix}.
\]

Clearly, when the system \( (A_a, \hat{A}_1, \hat{E}_a, \hat{E}_1) \) is detectable and stabilisable, a dynamic controller \( (A_c, B_c, C_c) \) exists to arbitrarily relocate the zeros, and consequently stabilise the zero dynamics. 

In this theorem, we gave a systematic controller design that moves the invariant zeros of a general stabilisable and detectable LTI plant to arbitrary locations. It is known that invariant zeros are invariant under feedback control (both state feedback and output feedback), and so our zero relocation through the use of new controller architecture is significant. Our design is based on the smart construction of a pre- + post- + feedforward compensator, which equivalences the zero-relocation problem with a feedback controller design problem directly from the zero dynamics. 

Let us make a couple of further comments about our design:

1. In essence, the estimation of \( x_{q-1} = \hat{y}^{(q-1)} \) from \( y \) (represented by the block D.E. in Figure 2) is straightforward, since the quantity to be estimated is part of the infinite-zero structure. As a note, one need not place the smoothing filter \( \frac{1}{1+(s + \epsilon)^q} \) in the feedforward path. In this case, we can use a time-scaling approach (Saberi and Sannuti 1988) to show that the fast dynamics introduced by the estimator only has a minor impact to the system dynamics (specifically, moving existing zeros only slightly and introducing highly stable zeros).

As an alternative to the high-gain estimator presented in the proof, a multiple-delay approximation can also be used for estimation (Roy, Saberi, and Wan 2010). It is worth noting that, for uniform-rank plants with relative degree 1, derivative estimation is not needed at all.

We notice that the approximation of output derivatives is a classical technique in traditional controller design. In designing such controllers
through approximation, performance measures such as robustness in the presence of noise need to be evaluated. We refer the readers to Roy et al. (2010) and Wan, Roy, Saberi, and Stoorvogel (2008, 2009) for more detailed discussion, and leave a careful development of robust designs to future work.

(2) The result presented in this article is applicable to many controller design problems. For instance, it permits stabilisation directly through the use of a high gain output-feedback control, which relies on the fact that the plant is minimum phase. Such a direct output-feedback methodology is especially valuable in the context of decentralised control, where the standard paradigm of estimation followed by state feedback fails (Wan et al. 2008, 2009). Zero relocation is also needed e.g. for adaptive control of non-minimum-phase plants and for plant-inverse controller design (Lozano-Leal 1989; Ortega 1990; Bayard and Boussalis 1993; Bayard 1994; Ortega et al. 1994).

In this communique, we have focused entirely on presenting the feedforward computation for zero relocation. In practice, of course a feedback controller design would be undertaken subsequent to the zero-relocation task, with the aim of shaping the closed-loop state dynamics of the virtual augmented plant (which includes the states of the actual plant). In many circumstances, the feedforward compensator and the feedback would, in practice, be implemented together as a single transfer function relating the plant output to the plant input.

3. Example

Control of teams of autonomous agents with distributed sensing and/or communication capabilities has been of wide interest in recent years, and has found application in such diverse domains as autonomous vehicle control and distributed computing in sensor networks (e.g. Wan et al. (2009)). The autonomous-agent control problem is a difficult one, and in particular it is recognised that control of non-minimum-phase devices in this setting is difficult. Here, let us consider a team containing $n$ identical agents, with the dynamics described by

$$
\begin{align*}
\dot{x} &= I \otimes \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} x + I \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \\
y &= I \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} x,
\end{align*}
$$

where $x \in \mathbb{R}^{2n}$, $u \in \mathbb{R}^n$ and $y \in \mathbb{R}^n$ are the state, input and output, respectively.

Clearly, the above system is square invertible, stabilisable, detectable and uniform rank 1. The above system can be transformed into the SCB form as

$$
\begin{align*}
\dot{x}_1 &= x_1, \\
x_1 &= -x_n + 2x_1 + u, \\
y &= x_1,
\end{align*}
$$

where $x_n \in \mathbb{R}^n$ is the state of the zero dynamics, and $x_1 \in \mathbb{R}^n$. The SCB form informs that the system has $n$ zeros at location 0, and hence is not minimum phase. Let us show how a feedforward compensator is designed to force the system to be minimum phase, and hence permit design of decentralised high-gain controllers to stabilise the system.

To do that, according to Theorem 1, we only need to find $A_c$, $B_c$, $C_c$ such that the poles of $\left[\begin{smallmatrix} 0 & -B_c \\ A_c & C_c \end{smallmatrix}\right]$ (the zeros of the system) are in the OLHP. For illustration, let us choose $C_c = 2I$, $B_c = 2I$, and $A_c = -12I$ to place the zeros at $-2$. The corresponding feedforward compensator is

$$
\begin{align*}
\dot{x}_c &= -12x_c + 2v, \\
y_c &= 2x_c, \\
u &= -24x_c + 3v, \\
\ddot{y} &= y - y_c,
\end{align*}
$$

where $x_c$, $y_c$, $v$ and $\ddot{y}$ are all $n \times 1$ vectors. For this new system with input $v$ and output $\ddot{y}$, the original system’s $n$ zeros and the extra $n$ zeros introduced by the feedforward compensator are all placed at $-2$. We note that the compensator design is decentralised in the sense that each agent constructs its own compensator locally. Since the resulting network is minimum phase and uniform rank, we can now apply decentralised high-gain controller to stabilise the system as shown in Wan et al. (2009).

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